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Exact deflection solutions of beams with shear piezoelectric actuators

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Abstract

Exact deflection models of beams with n actuators of shear piezoelectric are developed analytically. To formulate the models, the first-order and higher-order beam theories are used. The exact solutions are obtained with the aid of the state-space approach and Jordan canonical form. A case study is presented to evaluate the performance of the authors' previously reported models. Through a demonstrative example, a comparative study of the first-order and higher-order beams with two shear piezoelectric actuators is attained. It is shown that the first-order beam cannot predict the beam behavior when compared with the results of the higher-order beam. Further applications of the solutions are presented by investigating the effects of actuators lengths and locations on the deflected shapes of beams with two piezoelectric actuators. Some interesting deflection curves are presented. For example, the deflection curve of a H–H beam resembles saw teeth that rotate clockwise about the central location with the increase of actuators lengths. The presented exact solutions can be used in the design process to obtain detailed deformation information of beams with various boundary conditions. Moreover, the presented analysis can be readily used to perform precise shape control of beams with n actuators of shear piezoelectric.

Keywords: Piezoelectric; Beam; Shear

1. Introduction

The development of piezoelectric materials has constituted a revolution in sensing and actuation applications in recent years. The rapid response and high resolution, along with other properties such as large bandwidth and little power consumption, make piezoelectric materials increasingly popular as potential candidates for sensors and actuators substitution. Piezoelectric elements are being increasingly utilized in many structures including aerospace applications, sport goods, and MEMS applications. For instance, piezoelectric actuators are incorporated in flexible structures to provide precision position, to reject noise and vibration and to supply linear motion.

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When the piezoelectric actuators are employed in precise position control, precise deflection solution becomes necessary to predict the structure response. Analytical model can give exact information of the deformation. For example, the model can be used to obtain detailed deflection, slope and curvature change of a beam with piezoelectric actuators.

Recently, shear piezoelectric actuators have been used to generate deflection and to reject vibration in beams. The idea of exploiting the shear mode to create transverse deflection in sandwich beams was first suggested by Sun and Zhang (1995). A commercial finite element package was used to model numerically a cantilever beam with shear piezoelectric actuator. It was shown that embedded shear actuators offer many advantages over surface mounted extension actuators. In a more recent work, Zhang and Sun (1996) formulated an analytical model of a sandwich beam with shear piezoelectric actuator that occupies the entire core. The model derivation was simplified by assuming that the face layers follow Euler–Bernoulli beam, whereas the core layer obeys Timoshenko beam. Furthermore, a closed form solution of the static deflection was presented for a cantilever beam. A finite element approach was used by Benjeddou et al. (1997, 1999) to model a sandwich beam with shear and extension piezoelectric elements. The finite element model employed the displacement field of Zhang and Sun (1996). It was shown that the finite element results agree quite well with the analytical results. Raja et al. (2002) extended the finite element model of Benjeddou's research team to include vibration control scheme. It was observed that the shear actuator is more efficient in rejecting vibration than the extension actuator for the same control effort. Aldraihem and Khdeir (2000) proposed analytical models and exact solutions for beams with shear and extension piezoelectric actuators. The models are based on the first-order beam theory (FOBT) and higher-order beam theory (HOBT). The exact solutions are obtained by using the state-space approach along with Jordan canonical form. The deflections of beams with various boundary conditions were investigated. Vel and Batra (2001) presented an exact 3-D state space solution for the static cylindrical bending of simply supported plates with shear mode actuators.

The previous studies have been primarily devoted to two approaches; viz. analytical investigation of beams/plates with shear piezoelectric actuators occupying the entire core (Zhang and Sun, 1996; Aldraihem and Khdeir, 2000; Vel and Batra, 2001) and numerical investigation of cantilever beams with segmented shear piezoelectric actuators (Sun and Zhang, 1995; Benjeddou et al., 1997, 1999; Raja et al., 2002).

The present study is a continuation of the authors' recent work (Khdeir and Aldraihem, 2001), in which the deflections of first-order and higher-order beams with one shear piezoelectric actuator were obtained. Although the presented solutions in investigation (Khdeir and Aldraihem, 2001) were analytical, they were approximate in the sense of material properties. It was assumed that the stiffness of the sandwich beam is uniform and constant throughout the beam length. The aim of the present study is to develop analytical models and to present exact solutions of beams with n actuators of shear-mode piezoelectric. The first-order and higher-order beam theories will be applied in the formulation. The results of the proposed theories will be compared to demonstrate the effectiveness of each theory in predicting the beam deflection. The accuracy of the approximated solutions given in Khdeir and Aldraihem (2001) is evaluated via a comparison with the present exact solutions for beams with various boundary conditions. The deflections created by two shear actuators will be presented for beams with various boundary conditions. The effect of actuators length and location on the deflected shapes of the beam is also investigated.

2. Analytical formulation

Consider the sandwich beam shown in Fig. 1. The beam is composed of two facing layers and a central layer. The central layer is a core consisting of n piezoelectric actuators and foam segments. The actuators are shear-mode piezoelectric with poling direction along the x-direction. The foam segments, the actuators and the facing layers are perfectly bonded to each other. Each piezoelectric actuator can possess its own

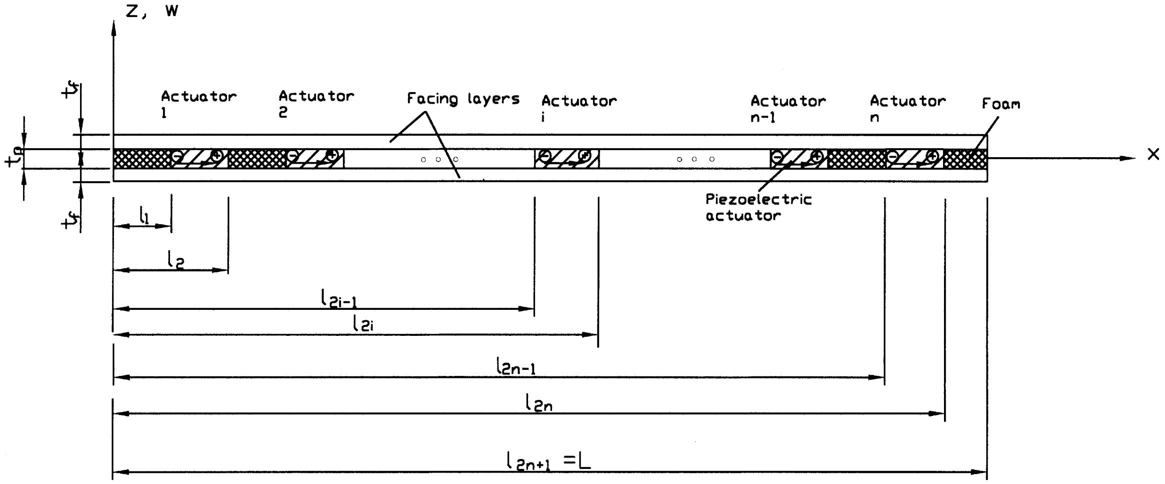

Fig. 1. The geometry of a beam with n actuators of shear piezoelectric.

material and geometrical properties. The sandwich beam is symmetric about its mid-plane and is not subjected to external mechanical loads. The material properties of the beam can be isotropic and specially orthotropic.

Using the principle of stationary potential energy, the governing equations of HOBT with shear piezoelectric actuators are derived as

$$\begin{aligned} &(M_x)' - n_1(P_x)' - Q_x + n_2 S_x = 0 \\ &n_1(P_x)'' + (Q_x)' - n_2(S_x)' = 0 \end{aligned} \tag{1}$$

with the following associated boundary conditions at $x = 0, L$:

$$\begin{array}{ll} \text{Essential B.C.} & \text{Natural B.C.} \\ \phi & M_x - n_1 P_x \\ w' & -n_1 P_x \\ w & n_1(P_x)' + Q_x - n_2 S_x \end{array} \tag{2}$$

where $n_1 = 4/3h^2$, $n_2 = 3n_1$ and h is the total thickness of the sandwich beam. A prime on a quantity denotes ordinary differentiation with respect to x.

The moment and force resultants in terms of displacement quantities are given by

$$\begin{aligned} M_x &= D_{11}\phi' - n_1 F_{11}(\phi' + w'') \\ P_x &= F_{11}\phi' - n_1 H_{11}(\phi' + w'') \\ Q_x &= (A_{55} - n_2 D_{55})(\phi + w') - Q_x^p \\ S_x &= (D_{55} - n_2 F_{55})(\phi + w') - S_x^p \end{aligned} \tag{3}$$

where w denotes the transverse displacement of the beam mid-plane and ϕ denotes the rotation of normal to the x-axis about the y-axis.

The beam stiffnesses are defined by

$$(D_{11}, F_{11}, H_{11}) = b\int_{-h/2}^{h/2}(z^2, z^4, z^6)\widetilde{Q}_{11}\,\mathrm{d}z$$
$$(A_{55}, D_{55}, F_{55}) = b\int_{-h/2}^{h/2}(1, z^2, z^4)\widetilde{Q}_{55}\,\mathrm{d}z \tag{4}$$

The piezoelectric stress resultants are expressed as

$$(Q_x^p, S_x^p) = b\int_{-h/2}^{h/2}(1, z^2)\widetilde{Q}_{55}E_1 d_{15}\,\mathrm{d}z \tag{5}$$

where

$$\widetilde{Q}_{11} = Q_{33} - \frac{Q_{23}^2}{Q_{22}}, \quad \widetilde{Q}_{55} = Q_{55} \tag{6}$$

and

$$Q_{ij} = c_{ij} - \frac{c_{i3}c_{j3}}{c_{33}}, \quad i,j = 1,2,4,5,6$$
$$Q_{i3} = c_{i3} - \frac{c_{1i}c_{13}}{c_{11}}, \quad i = 2,3 \tag{7}$$

where c_{ij} are the components of the stiffness matrix, d_{15} is the piezoelectric shear coefficient, E_1 is the electric field applied across the thickness of a shear piezoelectric actuator and b is the beam width.

Substituting Eq. (3) into (1) and perform some arrangements, the equilibrium equations can be written in terms of displacement quantities as

$$\phi'' = c_1(\phi + w') + c_2 w''' - f_1 Q_x^p + f_3 S_x^p$$
$$w'''' = c_3(\phi' + w'') + f_6(Q_x^p)' + f_7(S_x^p)' \tag{8}$$

where

$$c_1 = -\frac{e_1}{e_2}, \quad c_2 = -\frac{e_3}{e_2}, \quad c_3 = \frac{(e_1 + e_3c_1)}{(e_4 - e_3c_2)}$$
$$f_1 = \frac{1}{e_2}, \quad f_3 = \frac{n_2}{e_2}, \quad f_6 = \frac{(1 - e_3f_1)}{(e_4 - e_3c_2)}, \quad f_7 = \frac{(e_3f_3 - n_2)}{(e_4 - e_3c_2)}$$
$$e_1 = -A_{55} + 2n_2D_{55} - n_2^2F_{55}, \quad e_2 = D_{11} - 2n_1F_{11} + n_1^2H_{11}$$
$$e_3 = -n_1F_{11} + n_1^2H_{11}, \quad e_4 = -n_1^2H_{11} \tag{9}$$

The state space concept in conjunction with the Jordan canonical form will be used to analyze the deflection of sandwich beams with n shear piezoelectric actuators. Using this approach, Eqs. (8) will be written in the state form by introducing the following state variables:

$$Z_1 = \phi, \quad Z_2 = \phi', \quad Z_3 = w, \quad Z_4 = w', \quad Z_5 = w'', \quad Z_6 = w''' \tag{10}$$

Eqs. (8) along with Eqs. (10) can be combined into a single first-order system of equations as

$$\{Z'\} = [A]\{Z\} + \{F\} \tag{11}$$

where the matrix $[A]$ and the load vector $\{F\}$ are given by

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & c_1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & c_3 & 0 & 0 & c_3 & 0 \end{bmatrix}, \quad \{F\} = \begin{Bmatrix} 0 \\ f_3 S_x^p - f_1 Q_x^p \\ 0 \\ 0 \\ 0 \\ f_6 (Q_x^p)' + f_7 (S_x^p)' \end{Bmatrix} \tag{12}$$

Given that the matrix $[A]$ has an eigenvalue with multiplicity 4, the solution to (11) will be given in terms of the Jordan canonical form as

$$\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{Bmatrix} = \begin{cases} [M] e^{[J]x} \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{Bmatrix}, & 0 \leqslant x < l_1, \\ [M] e^{[J]x} \begin{Bmatrix} k_{12i-5} \\ k_{12i-4} \\ k_{12i-3} \\ k_{12i-2} \\ k_{12i-1} \\ k_{12i} \end{Bmatrix} + [M] e^{[J]x} \int^x e^{-[J]\eta} [M]^{-1} \{F(\eta)\} \, \mathrm{d}\eta, & l_{2i-1} < x < l_{2i}, \qquad i = 1, \ldots, n \\ [M] e^{[J]x} \begin{Bmatrix} k_{12i+1} \\ k_{12i+2} \\ k_{12i+3} \\ k_{12i+4} \\ k_{12i+5} \\ k_{12i+6} \end{Bmatrix}, & l_{2i} < x < l_{2i+1}, \end{cases} \tag{13}$$

where $[M]$ is a modal matrix, which contains eigenvectors and generalized eigenvectors of the matrix $[A]$, and $e^{[J]x}$ is a block diagonal defined as

$$e^{[J]x} = \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & 0 & 0 \\ 0 & 1 & x & \frac{1}{2}x^2 & 0 & 0 \\ 0 & 0 & 1 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda x} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\lambda x} \end{bmatrix} \tag{14}$$

where

$$\lambda = \sqrt{c_1 + c_3 + c_2 c_3} \tag{15}$$

Substituting $12n$ continuity conditions and three boundary conditions at each end ($x = 0$, $x = L$) for the desired combinations of boundary conditions, one has to solve $12n + 6$ simultaneous equations to find the constants $k_1, k_2, \ldots, k_{12n+6}$. A similar procedure can be followed to analyze the deflection using the FOBT, the reader should refer to Khdeir and Aldraihem (2001). In the FOBT, one needs $8n$ continuity conditions and two boundary conditions at each end for the desired combinations of boundary conditions to determine the k's constants.

3. Results and discussion

The exact solutions developed in the previous section will be used to investigate the deflection of beams with hinged–hinged (H–H), clamped–hinged (C–H), clamped–clamped (C–C), and clamped–free (C–F) boundary conditions. The following materials properties are used in the analysis:

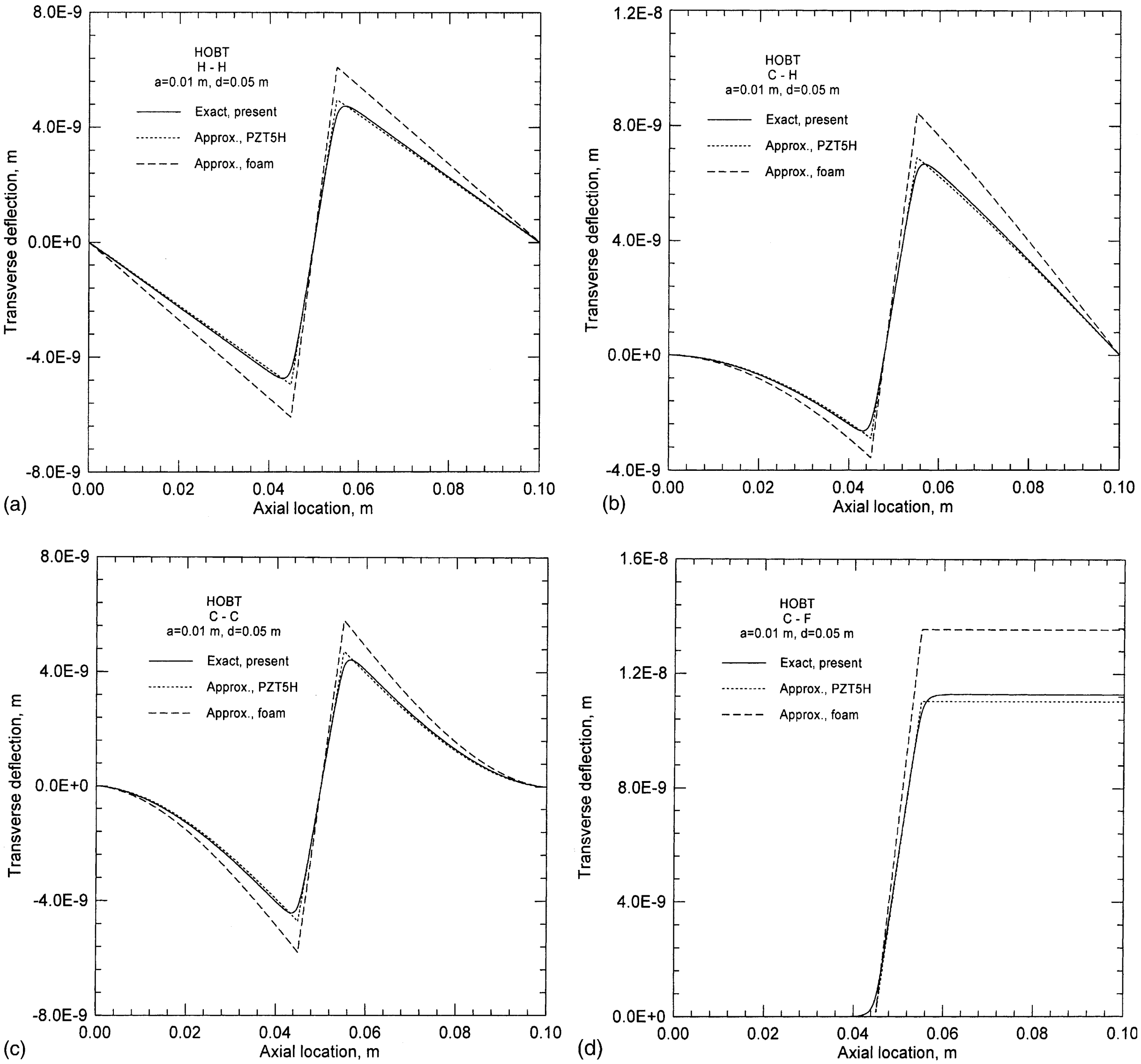

Fig. 2. The deflection curves of the HOBT: exact versus approximate for (a) H–H, (b) C–H, (c) C–C, (d) C–F boundary conditions.

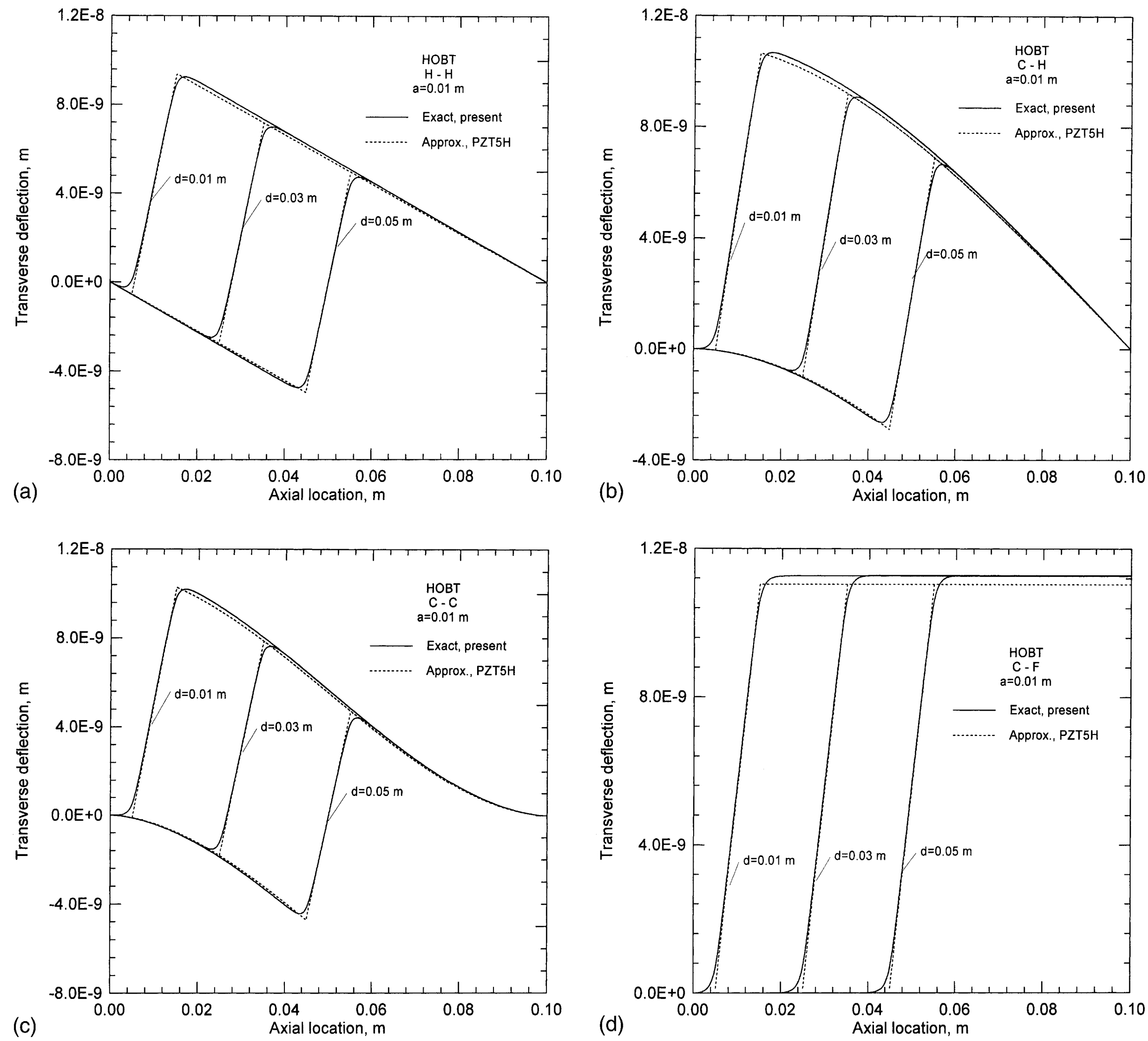

Fig. 3. The effect of the actuator central location on the deflection of the HOBT: exact versus approximate for (a) H–H, (b) C–H, (c) C–C, (d) C–F boundary conditions.

Aluminum (Khdeir and Aldraihem, 2001)

$$E = 70.3 \text{ GPa}, \quad \nu = 0.34$$

Foam (Khdeir and Aldraihem, 2001)

$$E = 35.3 \text{ MPa}, \quad G = 12.76 \text{ MPa}$$

PZT5H (*Electro Ceramic Division*) [1]

[1] Electro Ceramic Division, Data for Designers, Morgan Matroc Inc., 232 Forbes Road, Bedford, OH 44146.

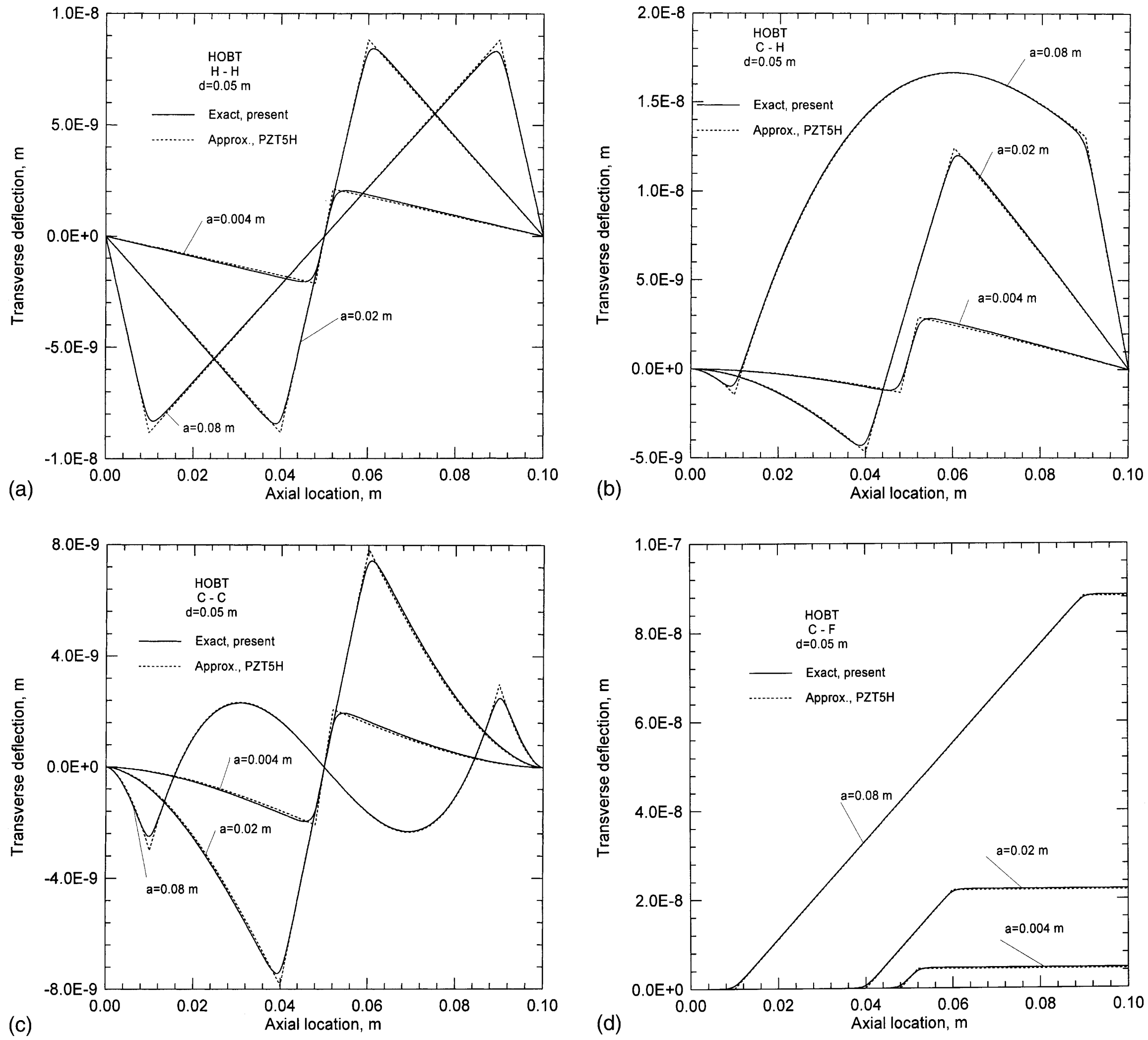

Fig. 4. The effect of the actuator length on the deflection of the HOBT: exact versus approximate for (a) H–H, (b) C–H, (c) C–C, (d) C–F boundary conditions.

$$c_{11} = 126 \text{ GPa}, \quad c_{12} = 79.5 \text{ GPa}, \quad c_{13} = 84.1 \text{ GPa}, \quad c_{33} = 117 \text{ GPa}, \quad c_{44} = 23 \text{ GPa}$$

$$d_{15} = 739.13 \times 10^{-12} \text{ m/V}$$

The geometric configuration of the beam, as shown in Fig. 1, is assumed to be

$$L = 0.1 \text{ m}, \quad t_f = 0.008 \text{ m}, \quad t_P = 0.002 \text{ m}$$

The piezoelectric actuators are assumed to have identical material properties and to have same poling orientations. The voltage applied on each actuator is 20 V. In the FOBT, a value of 5/6 was taken for the shear correction factor.

3.1. Comparison with approximate results

The exact analytical solutions presented in the previous section are used to evaluate the accuracy and the validity of the approximate analytical solutions (Khdeir and Aldraihem, 2001). The deflection curves of beams with one PZT5H actuator are obtained. The effects of actuator central location and the length on the results are examined. In study (Khdeir and Aldraihem, 2001), the beam stiffnesses (A_{55}, D_{11}, D_{55}, F_{11}, F_{55}, H_{11}) are assumed to be constant along the beam axis. To calculate these stiffnesses, the whole core is assumed to have the properties of the foam.

Fig. 2 shows the beam deflection obtained by the present exact solution and the approximate solution (Khdeir and Aldraihem, 2001). In the approximate analysis, two sets of deflection results are determined. One set of results is for beams with cores of foam properties, and another set of results is for beams with cores of PZT5H properties. For H–H, C–H and C–C beams, the approximate results of beams with PZT5H core properties agree quit well with those obtained by the present exact results. If the beam core is assumed to possess the foam properties, the approximate analysis overestimates the deflection when compared to that of the exact analysis. The largest disagreement between the exact and the approximate results appears in the C–F beams.

The exact and the approximate analyses are further compared by examining the effects of the actuator central location ($d = (l_2 + l_1)/2$) and length ($a = l_2 - l_1$) on the results. In the approximate predication only beams with PZT5H core properties are considered since the results of this approximation are shown above to be closer to those of the exact results. Figs. 3 and 4 show the effects of the actuator location and length, respectively, on the deflected shape. Changing the location of the piezoelectric actuator gives slight difference between the deflection of the approximate and exact analysis. The difference increases, as the actuator gets closer to the beam boundaries, except for C–F beam. Away from the actuator ends, increasing the length of the actuator reduces the difference between the results of the approximate and the exact analyses.

3.2. Beams with two actuators

Beams with two shear piezoelectric actuators are considered. The exact model will be used in the analysis. First, the validity of the FOBT results is evaluated for beams with two actuators of identical lengths ($a_1 = a_2 = 0.01$ m) and located at $d_1 = (l_1 + l_2)/2 = 0.025$ m and $d_2 = (l_3 + l_4)/2 = 0.075$ m. Fig. 5

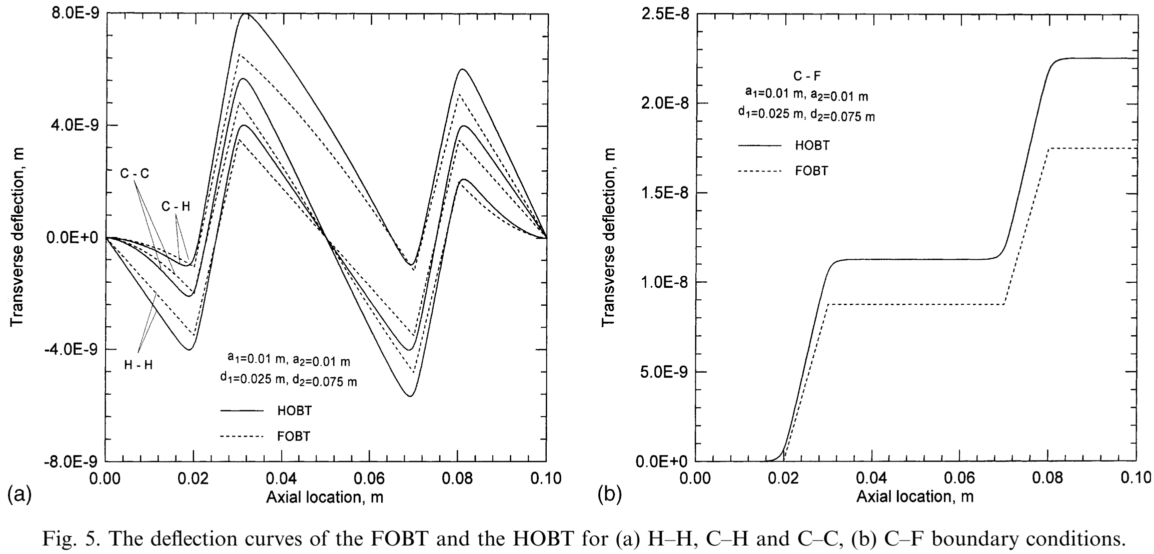

Fig. 5. The deflection curves of the FOBT and the HOBT for (a) H–H, C–H and C–C, (b) C–F boundary conditions.

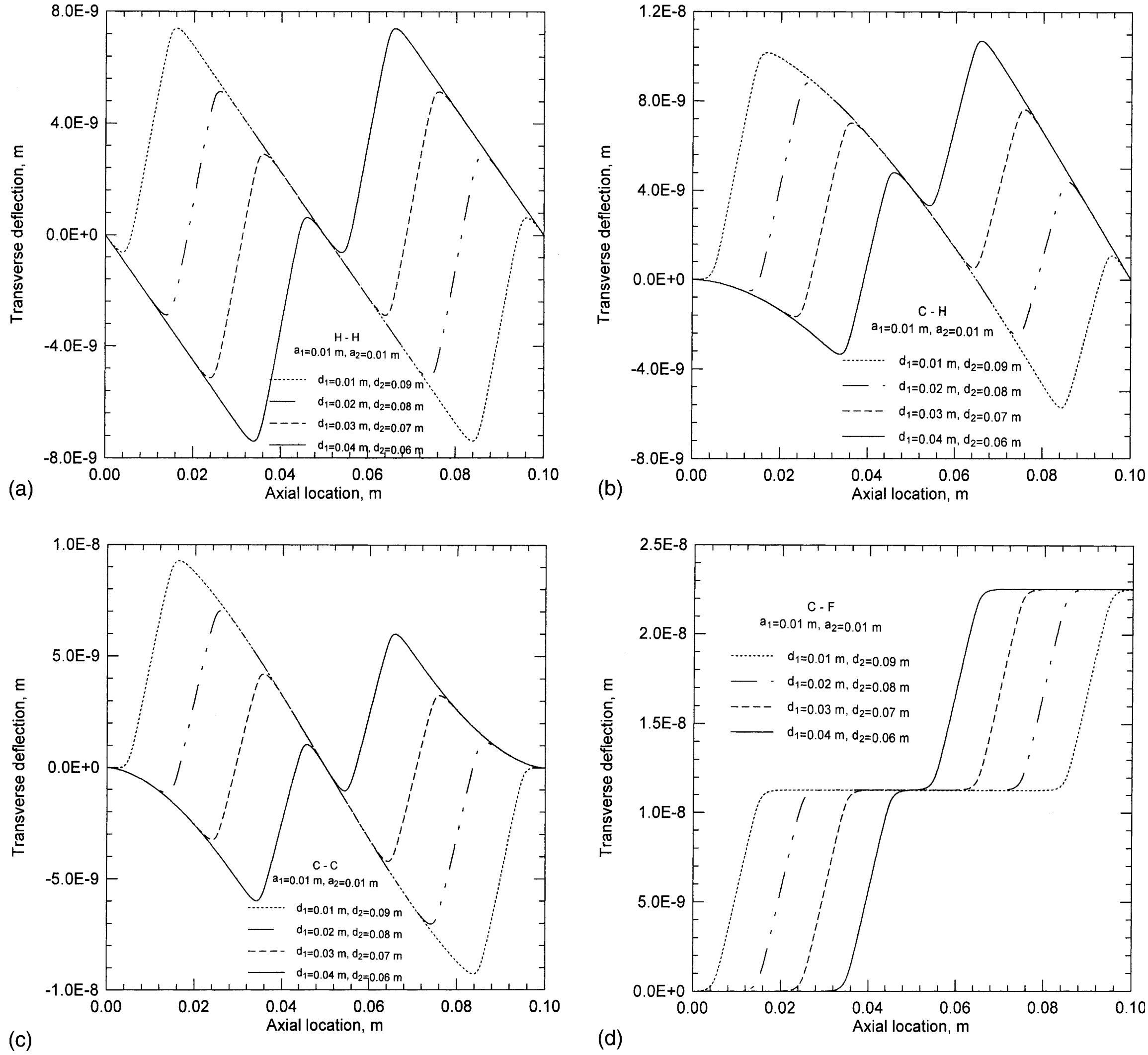

Fig. 6. The effect of the actuators central locations (d_1 and d_2) on the deflection of beams with (a) H–H, (b) C–H, (c) C–C, (d) C–F boundary conditions.

shows the deflected shapes of the FOBT and the HOBT. It is shown that there is a pronounced difference between the results of the FOBT and to the HOBT. For all boundary conditions, the FOBT underestimate the deflection when compared with the results of the HOBT. These results indicate that the first-order beams are stiffer than the higher-order beams.

Since the FOBT lacks the ability to accurately predict the beam behavior, the HOBT will be used next to investigate the effects of the actuator location and length on the deflection. Fig. 6 presents the effect of the actuators central locations (d_1 and d_2) on the deflection of beams with various boundary conditions. The length of the piezoelectric actuators are fixed to $a_1 = a_2 = 0.01$ m. For H–H and C–H beams, the absolute maximum deflection increases by moving the actuators toward the beam center or toward the supports. For C–C beams, the absolute maximum deflection increases by moving the actuators toward

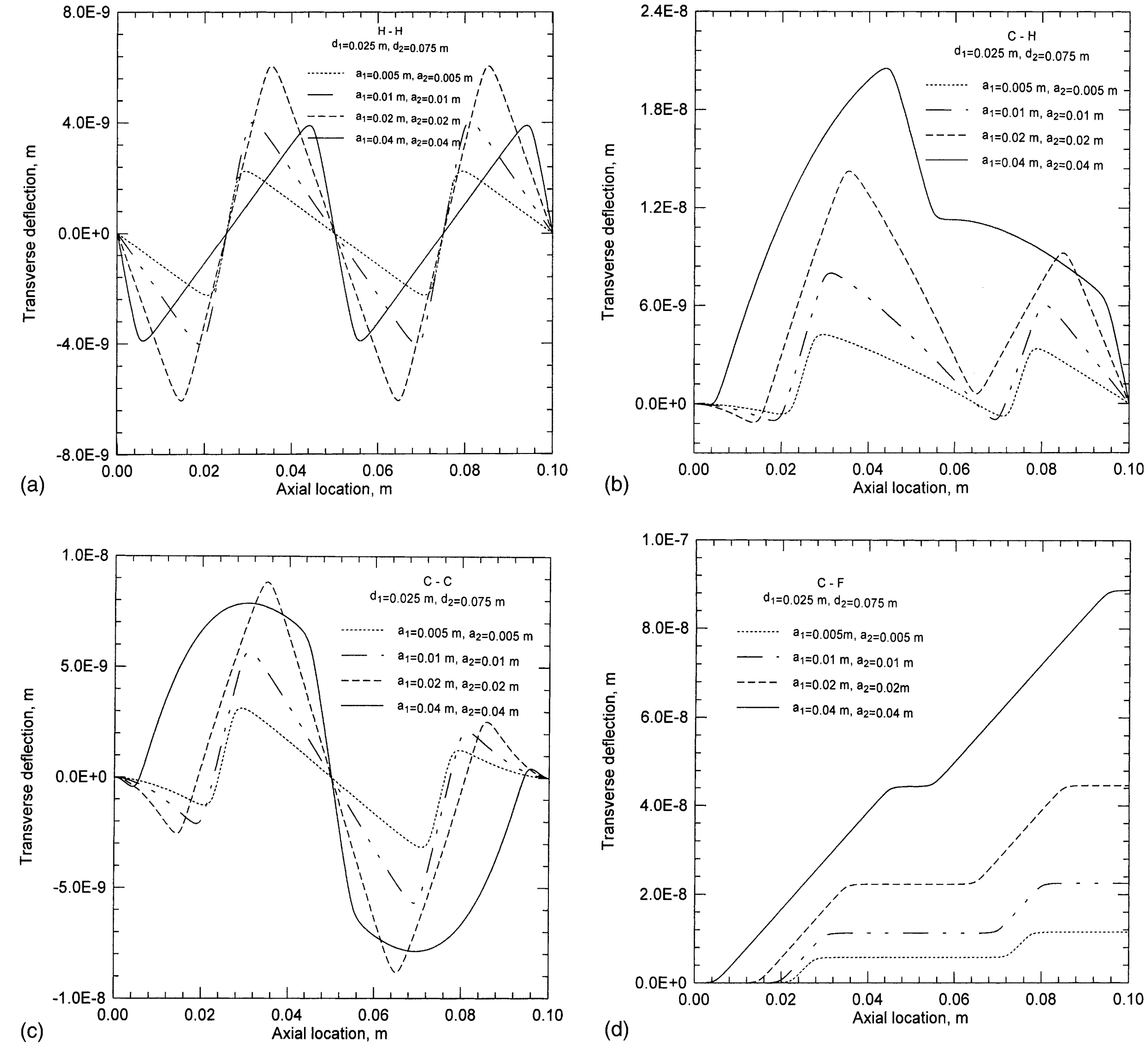

Fig. 7. The effect of the actuators lengths (a_1 and a_2) on the deflection of beams with (a) H–H, (b) C–H, (c) C–C, (d) C–F boundary conditions.

the beams ends. For C–F beams, altering the central locations has no influence on the tip displacements of the beams.

Fig. 7 shows the effect of the actuators lengths on the deflected shape. The central location of actuator 1 is fixed at $d_1 = 0.025$ m and of actuator 2 is fixed at $d_2 = 0.075$ m. Four significant values of a_1 and a_2 are studied. For H–H beams, the deflection curve resembles saw teeth that rotate clockwise about the central location with the increase of actuators lengths. Moreover, the slope of deflection becomes steeper in the actuator region if the actuator length is reduced. In C–H beams, the deflection increases when the lengths of the actuators are increased. The results of C–C beams are interesting. For the considered parameters, increasing the actuators lengths does not necessarily increase the maximum deflection. In fact, actuators of lengths $a_1 = a_2 = 0.02$ m can generate maximum deflection greater than that produced by actuators of lengths $a_1 = a_2 = 0.04$ m. For C–F beams, the deflection increases when the actuators lengths are increased.

4. Conclusions

Exact analytical solutions are presented for beams with n actuators of shear piezoelectric. The solutions obey the first-order and higher-order beam theories. The exact analytical solutions presented in this work are used to evaluate the validity of the authors' previously reported solutions (Khdeir and Aldraihem, 2001). For the considered sandwich beams and when the core is assumed to have the properties of PZT5H, the results of the approximate analysis (Khdeir and Aldraihem, 2001) agree quit well with those of the present exact results. Changing the location of the actuator gives slight difference between the results of the approximate and exact analyses. However, increasing the length of the actuator reduces the difference between the results of the approximate and the exact analysis. The approximate analysis has one advantage over the exact analysis. The approximate analysis requires solving only six simultaneous equations irrespective of the number of actuators and the type of boundary conditions. On the other hand, the exact analysis requires solving $12n + 6$ simultaneous equations to obtain the solution for a beam with n piezoelectric actuators.

Through a demonstrative example, a comparative study of the first-order and higher-order beams with two actuators is attained. For the considered beam's properties, it is observed that the first-order beam cannot predict the beam behavior when compared with the results of the higher-order beam.

Further applications of the solutions are presented by investigating the effects of actuators lengths and locations on the deflected shape of beams with two shear actuators. Some interesting deflected shapes are presented. For example, the deflection curve of a H–H beam resembles saw teeth that rotate clockwise about the central location with the increase of actuators lengths.

The presented solutions can be used in the design process to obtain detailed deformation information of beams with various boundary conditions. Moreover, the presented exact analysis can be readily used to perform precise shape control of beams with n segments of shear piezoelectric. Finally, the present analysis can be used as benchmarks for approximate solutions obtained by the finite element method and the Rayleigh–Ritz method.

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